

# VANISHING THEOREM FOR IRREDUCIBLE SYMMETRIC SPACES OF NONCOMPACT TYPE

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ABSTRACT. We prove the following vanishing theorem. Let  $M$  be an irreducible symmetric space of noncompact type whose dimension exceeds 2 and  $M \neq SO_0(2, 2)/SO(2) \times SO(2)$ . Let  $\pi : E \rightarrow M$  be any vector bundle, Then any  $E$ -valued  $L^2$  harmonic 1-form over  $M$  vanishes. In particular we get the vanishing theorem for harmonic maps from irreducible symmetric spaces of noncompact type.

## 1. INTRODUCTION

It was conjectured by J. H. Sampson [SAM] that any harmonic map with finite energy from a complete simply connected Riemannian manifold with negative sectional curvature whose dimension exceeds 2 must be constant. This is valid for space forms, but unsolved in general case. For Cartan-Hadamard manifolds, In [XIN1] Xin proved a general vanishing theorem as follows.

**Theorem 1.1.** *Let  $M$  be an  $m$ -dimensional Cartan-Hadamard manifold with the sectional curvature  $-a^2 \leq K \leq 0$  and the Ricci curvature bounded from above by  $-b^2$ . Let  $f$  be a harmonic map from  $M$  into any Riemannian manifold with the moderate divergent energy. If  $b \geq 2a$ , then  $f$  has to be constant.*

For Hermitian symmetric spaces of noncompact type, In [XIN2] Xin proved the following results.

**Theorem 1.2.** *A harmonic map of a finite energy from a classical bounded symmetric domain except  $D_{IV}(2) = SO_0(2, 2)/SO(2) \times SO(2) (\cong \mathbb{H} \times \mathbb{H})$  to any Riemannian manifold has to be constant.*

Now in [LIU] by calculating the lower bounds of sectional curvature of all irreducible symmetric spaces of noncompact type, we get the following theorem by using Theorem 1.1.

**Theorem 1.3.** *Let  $M$  be one of the irreducible symmetric spaces of noncompact type in the following cases,*

$$\begin{aligned}
& SL(n, \mathbb{R})/SO(n), n \geq 4; \\
& SU^*(2n)/Sp(n); \\
& SU(p, q)/S(U_p \times U_q), p + q \geq 4; \\
& SO_o(p, q)/SO(p) \times SO(q), \text{ for } r = 1, p + q \geq 4, \\
& \quad \text{for } r > 1, p + q \geq 6, \quad \text{here } r = \min(p, q); \\
& SO^*(2n)/U(n), n \geq 3; \\
& Sp(n, \mathbb{R})/U(n), n \geq 3; \\
& Sp(p, q)/Sp(p) \times Sp(q); \\
& EI, EII, EIII, EIV, EV, EVI, EVII, EVIII, EIX, FI, FII \text{ and } G.
\end{aligned}$$

*Then any  $L^2$  harmonic 1-form vanishes.*

In this article we prove the following theorem.

**Theorem 1.4.** *Let  $M$  be an irreducible symmetric space of noncompact type whose dimension exceeds 2 and  $M \neq SO_0(2, 2)/SO(2) \times SO(2)$ . Let  $\pi : E \rightarrow M$  be any vector bundle, Then any  $E$ -valued  $L^2$  harmonic 1-form over  $M$  vanishes.*

**Remark 1.1.** *Let  $f : M \rightarrow N$  be a harmonic map, then  $df \in \Lambda^1 f^* TN$  is a harmonic 1-form over  $M$ , from Theorem 1.4 we get the vanishing theorem for harmonic maps from irreducible symmetric spaces of noncompact type which generalize Theorem 1.1 and 1.2.*

In section 2 we prove a vanishing theorem for harmonic forms. In section 3 we give the vanishing theorem for Riemannian symmetric spaces of noncompact type. In section 4 we calculate the Hessian of distance function in the cases of Riemannian symmetric spaces of noncompact type and give the proof of Theorem 1.4.

## 2. THE VANISHING THEOREM OF HARMONIC FORMS

Let  $M$  be an  $n$  dimensional complete Riemannian manifold, let  $\pi : E \rightarrow M$  be a real vector bundle with rank  $r$ , we denote  $\nabla$  the metric connection on  $E$  which gives rise to the Levi-Civita connection when restricted to  $M$ .

Let  $\{e_i, 1 \leq i \leq n\}$  be an local orthogonal frame on  $M$  with dual coframe fields  $\{\theta^i\}$ . We identify the tangent vector field  $X$  with a 1-form  $\hat{X}$  via Riemannian inner product by

$$\hat{X}(Y) = \langle Y, X \rangle.$$

Then  $\hat{e}_i = \theta^i$ . In the following calculations, we take the normal coordinate at the given point  $x \in M$ , i.e.,  $\nabla(e_i)(x) = 0$ .

Let  $\wedge^p(M, E)$  be the vector space of all  $E$ -valued  $p$ -forms on  $M$ , for  $\omega \in \wedge^p(M, E)$ , we have exterior multiplication and interior product operator

$$\begin{aligned}\varepsilon_X(\omega) &= \varepsilon_{\hat{X}}(\omega) = \hat{X} \wedge \omega, \\ i_X(\omega) &= i_{\hat{X}}(\omega), \quad i_X(\omega)(X_1, \dots, X_{p-1}) = \omega(X, X_1, \dots, X_{p-1}).\end{aligned}$$

We have the following commutation rules

$$\varepsilon_X \varepsilon_Y + \varepsilon_Y \varepsilon_X = 0.$$

$$i_X i_Y + i_Y i_X = 0.$$

$$\varepsilon_X i_Y + i_Y \varepsilon_X = \langle X, Y \rangle 1, \text{ where } 1 \text{ is the identity operator.}$$

$$\nabla_X \varepsilon_Y = \varepsilon_Y \nabla_X + \varepsilon_{\nabla_X Y}.$$

$$\nabla_X i_Y = i_Y \nabla_X + i_{\nabla_X Y}$$

We can define the adjoint operator with respect to the inner product on  $\wedge^*(M, E)$ , then

$$\begin{aligned}\varepsilon_k^* &= \varepsilon_{e_k}^* = i_k = i_{e_k} \\ \nabla_k^* &= \nabla_{e_k}^* = -\nabla_k\end{aligned}$$

Now we have the exterior differential operator

$$d : \wedge^p(M, E) \rightarrow \wedge^{p+1}(M, E), \quad d = \varepsilon_k \nabla_k = \varepsilon_{\theta^k} \nabla_{e_k}.$$

its adjoint operator is

$$\delta = d^* = -i_k \nabla_k.$$

Let  $X$  be a tangent vector field, we have the Lie derivative with respect to  $X$

$$L_X = d \circ i_X + i_X \circ d.$$

We have

$$\begin{aligned}d \circ i_X &= \varepsilon_k \nabla_k i_X = \varepsilon_k (i_X \nabla_k + i_{\nabla_k X}) \\ &= (-i_X \varepsilon_k + \langle X, e_k \rangle) \nabla_k + \varepsilon_k i_{\nabla_k X} \\ &= -i_X \varepsilon_k \nabla_k + \nabla_X + \varepsilon_k i_{\nabla_k X} \\ i_X \circ d &= i_X \varepsilon_k \nabla_k \\ L_X &= \nabla_X + \varepsilon_k i_{\nabla_k X}.\end{aligned}$$

In particular, let  $f(x)$  be a function on  $M$ , we denote  $X = \nabla f(x)$  its gradient vector field. The Hessian of  $f$  is

$$Hess(f) = h_{kl} \theta^k \otimes \theta^l = h_{kl} e_k \otimes e_l.$$

Then

$$\nabla_k \nabla f = h_{kl} e_l, L_{\nabla f} = \nabla_{\nabla f} + h_{kl} \varepsilon_k i_l.$$

It follows that

$$\begin{aligned}
\langle L_{\nabla f}\omega, \omega \rangle &= \langle \nabla_{\nabla f}\omega, \omega \rangle + h_{kl} \langle \varepsilon_k i_l \omega, \omega \rangle \\
&= \frac{1}{2} \nabla_{\nabla f} |\omega|^2 + h_{kl} \langle i_k \omega, i_l \omega \rangle \\
&= \frac{1}{2} \langle \nabla f, \nabla |\omega|^2 \rangle + H_f(\omega, \omega).
\end{aligned}$$

where

$$H_f(\omega, \omega) = \sum_{kl} h_{kl} \langle i_k \omega, i_l \omega \rangle = \sum \langle h_{kl} \varepsilon_k i_l \omega, \omega \rangle.$$

We recall the Green formula, let  $D \subset M$  be a relatively compact domain with smooth boundary, we denote  $\nu$  the unit outward normal vector field on boundary. We have

$$\begin{aligned}
&\int_D [\langle d\omega, \psi \rangle - \langle \omega, \delta\psi \rangle] \\
&= \int_{\partial D} \langle \omega, i_\nu \psi \rangle, \quad \omega \in \wedge^p(M, E), \psi \in \wedge^{p+1}(M, E).
\end{aligned}$$

We calculate the integral in two ways

$$\begin{aligned}
\int_D \langle L_{\nabla f}\omega, \omega \rangle &= \int_D \langle d \circ i_{\nabla f}\omega + i_{\nabla f} \circ d\omega, \omega \rangle \\
&= \int_D \langle i_{\nabla f}\omega, \delta\omega \rangle + \int_{\partial D} \langle i_{\nabla f}\omega, i_\nu \omega \rangle \\
&\quad + \int_D \langle d\omega, \varepsilon_{\nabla f}\omega \rangle \\
\int_D \langle \nabla f, \nabla |\omega|^2 \rangle &= \int_D \operatorname{div}(|\omega|^2 \nabla f) - \Delta f |\omega|^2 \\
&= - \int_D \Delta f |\omega|^2 + \int_{\partial D} \nabla_\nu f |\omega|^2.
\end{aligned}$$

We get

$$\begin{aligned}
&\int_D \left[ -\Delta f \frac{|\omega|^2}{2} + H_f(\omega, \omega) - \langle i_{\nabla f}\omega, \delta\omega \rangle - \langle d\omega, \varepsilon_{\nabla f}\omega \rangle \right] \\
&= \int_{\partial D} \left[ \langle i_{\nabla f}\omega, i_\nu \omega \rangle - \frac{|\omega|^2}{2} \nabla_\nu f \right]
\end{aligned}$$

in other words,

$$\begin{aligned}
&\int_D [\Delta f |\omega|^2 - 2H_f(\omega, \omega) + 2 \langle i_{\nabla f}\omega, \delta\omega \rangle + 2 \langle d\omega, \varepsilon_{\nabla f}\omega \rangle] \\
&= \int_{\partial D} [-2 \langle i_{\nabla f}\omega, i_\nu \omega \rangle + |\omega|^2 \nabla_\nu f]
\end{aligned}$$

Let  $o \in M$  be a fixed point and  $r(x) = \operatorname{dist}(o, x)$  the distance function from  $o$ . We denote  $B_r = B(o, r) = \{x \in M \mid r(x) < r\}$ ,  $S_r = \partial B_r =$

$\{x \in M | r(x) = r\}$ . The Hessian of  $r(x)$  is the second fundamental form with respect to  $S_r$ ,

$$Hess(r)(X, Y) = \langle \nabla_X \frac{\partial}{\partial r}, Y \rangle, \quad X, Y \in TS_r(x).$$

Let  $\lambda_1(x) \geq \lambda_2(x) \cdots \geq \lambda_{n-1}(x)$  be the  $n-1$  eigenvalues of  $Hess(r)$ , we get the Laplacian of  $r$ , i.e.,

$$\Delta r = \text{tr}Hess(r) = \sum_{1 \leq i \leq n-1} \lambda_i(x).$$

**Theorem 2.1.** *Let  $M$  be a noncompact complete Riemannian manifold,  $0 \leq p \leq n$  be an integer, let  $\pi : E \rightarrow M$  be a vector bundle. Let  $o \in M$  be a fixed point with the distance function  $r(x) = \text{dist}(o, x)$ . We order the eigenvalues of  $Hess(r)$  with multiplicities in the way that  $\lambda_1(x) \geq \lambda_2(x) \cdots \geq \lambda_{n-1}(x)$ . If there exists  $R_0 \geq 0$ , we have for  $r(x) \geq R_0$ ,*

$$(2.1) \quad \sum_{i=1}^p \lambda_i(x) \leq \sum_{i=p+1}^{n-1} \lambda_i(x).$$

Then any  $E$ -valued  $L^2$   $p$ -harmonic form on  $M$  vanishes.

*Proof.* Let  $\omega \in \wedge^p(M, E)$ , let  $f(x) = r(x)$ ,  $\nu = \nabla r$ . Since  $d\omega = \delta\omega = 0$ , From (2.1) we get

$$\begin{aligned} \int_D [\Delta r |\omega|^2 - 2H_r(\omega, \omega)] &= \int_{\partial D} [-2 \langle i_\nu \omega, i_\nu \omega \rangle + |\omega|^2 \nabla_\nu r] \\ &\leq \int_{\partial D} |\omega|^2 \end{aligned}$$

At  $x$  we choose normal frame  $\{e_i\}$  with  $e_n = \nu = \nabla r$ ,  $e_i \in TS_r$ ,  $1 \leq i \leq n-1$ , we suppose  $Hess(r)$  is diagonal with respect to  $\{e_i\}$ , then

$$\begin{aligned} Hess(r) &= \sum_{1 \leq i \leq n-1} \lambda_i(x) e_i \otimes e_i. \\ \Delta r &= \sum_{1 \leq i \leq n-1} \lambda_i(x). \\ Hess_r(\omega, \omega) &\leq \sum_{1 \leq i \leq p} \lambda_i \langle \omega, \omega \rangle. \end{aligned}$$

We get

$$\begin{aligned} \Delta r |\omega|^2 - Hess_r(\omega, \omega) &\geq [\frac{1}{2} \Delta r - \sum_{1 \leq i \leq p} \lambda_i(x)] |\omega|^2 \\ &= \frac{1}{2} [\sum_{p+1 \leq i \leq n-1} \lambda_i(x) - \sum_{1 \leq i \leq p} \lambda_i(x)] |\omega|^2 \end{aligned}$$

Under the condition (2.1), we take  $D = B_{R_2} \setminus B_{R_1}$ ,  $R_0 \leq R_1 < R_2$ , then

$$0 \leq \int_{S_{R_2}} |\omega|^2 - \int_{S_{R_1}} |\omega|^2.$$

We see that  $\int_{S_R} |\omega|^2$  is non decreasing with respect to  $R$  for  $R \geq R_0$ . The unique continuation theorem of harmonic forms says that the harmonic form  $\omega$  must vanishes anywhere when  $\omega$  vanishes in some open subset of  $M$ . We see that if  $\omega \neq 0$ , then there exists  $R \geq R_0$  such that  $\int_{S_R} \frac{1}{2} |\omega|^2 \geq c > 0$ , which gives rise to

$$\frac{1}{2} \int_M |\omega|^2 \geq \int_R^\infty \left( \int_{S_r} \frac{1}{2} |\omega|^2 \right) dr \geq \int_R^\infty c dr = \infty.$$

this contradicts to the condition of finite energy.  $\square$

**Remark 2.1.** *In fact we can get the vanishing theorem if we have the estimate  $\Delta r |\omega|^2 - \text{Hess}_r(\omega, \omega) \geq 0$  outside some compact subset of  $M$ , We can give the similar proof by using the conservation law of stress-energy in [JX]. The formula (2.1) is given at [EF].*

### 3. THE VANISHING THEOREM IN THE CASE OF RIEMANNIAN SYMMETRIC SPACES

Let  $M$  be a simply connected Riemannian symmetric space of non compact type with dimension  $n$ . Let  $o \in M$  be a fixed point with the distance function  $r(x) = \text{dist}(o, x)$ .  $\gamma(t) : [0, r(x)] \rightarrow M$  be the unique geodesic from  $o$  to  $x$ . The curvature transformation along  $\gamma$  is the self adjoint operator

$$\begin{aligned} \mathcal{R}_{\dot{\gamma}(t)} V &= R(V, \dot{\gamma}(t))\dot{\gamma}(t), \\ < \mathcal{R}_{\dot{\gamma}(t)} V, W > &= R(\dot{\gamma}(t), V, \dot{\gamma}(t), W), \forall V, W \perp \dot{\gamma}(t). \end{aligned}$$

We choose the orthogonal frame  $\{e_i, 1 \leq i \leq n\}$  along  $\gamma$  so that  $e_n(t) = \dot{\gamma}(t)$ . Let  $-\lambda_i^2(t), 1 \leq i \leq n-1 \geq 0$  be the eigenvalues of  $\mathcal{R}_{\dot{\gamma}(t)}$ , we order  $\lambda_i(t)$  with multiplicities in the form of  $\lambda_1(t) \geq \lambda_2(t) \geq \dots \geq \lambda_{n-1}(t)$ . Since the curvature tensor of  $M$  is parallel, all  $\lambda_i(t)$  are constant along  $\gamma$ ,  $\lambda_i(t) \equiv \lambda_i(0) = \lambda_i$ . At  $x$ , we have

$$\begin{aligned} \Delta r(x) &= \sum_{i=1}^{n-1} \lambda_i \coth(\lambda_i r) \\ \text{Hess}(r) &= \sum_{i=1}^{n-1} \lambda_i \coth(\lambda_i r) e_i(r) \otimes e_i(r) \\ \text{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) &= \text{Ric}(e_n, e_n) = - \sum_{i=1}^{n-1} \lambda_i^2 \end{aligned}$$

**Theorem 3.1.** *Let  $M$  be a simply connected Riemannian symmetric space of non compact type, let  $\pi : E \rightarrow M$  be a vector bundle. let  $0 \leq p \leq n$  be an integer. If at some point  $o \in M$  (hence at all points), the eigenvalues of curvature transformation operator  $\mathcal{R}_v, v \in T_o M$  be any unit vector at  $o$ , which ordered by  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1}$  satisfy*

$$(3.1) \quad \sum_{i=1}^p \lambda_i \leq \sum_{i=p+1}^{n-1} \lambda_i.$$

*Then any  $L^2$   $E$ -valued harmonic  $p$ -forms vanish. In particular, let the Ric curvature be  $Ric = -B$ , the curvature lower bounds of  $M$  is  $-A$ , if*

$$(3.2) \quad p(p+1) \leq \frac{B}{A},$$

*then any  $L^2$  harmonic  $p$ -form vanishes.*

*Proof.* If (3.1) holds, as  $\phi(\lambda) = \coth(\lambda r)$  is decreasing with respect to  $\lambda$ , we have

$$\sum_{p+1 \leq i \leq n-1} \lambda_i \coth(\lambda_i r) - \sum_{1 \leq i \leq p} \lambda_i \coth(\lambda_i r) \geq 0$$

Then if (3.1) holds, from Theorem 2.1, we get any  $L^2$  harmonic form vanish.

If (3.2) holds, we have

$$\sum_{1 \leq i \leq n-1} \lambda_i^2 \geq p(p+1)\lambda_1^2 \geq (p+1) \sum_{1 \leq i \leq p} \lambda_i^2.$$

i.e.,

$$\sum_{p+1 \leq i \leq n-1} \lambda_i^2 \geq p \sum_{1 \leq i \leq p} \lambda_i^2 \geq \left( \sum_{1 \leq i \leq p} \lambda_i \right)^2.$$

We get

$$\left( \sum_{p+1 \leq i \leq n-1} \lambda_i \right)^2 \geq \sum_{p+1 \leq i \leq n-1} \lambda_i^2 \geq \left( \sum_{1 \leq i \leq p} \lambda_i \right)^2.$$

Then (3.1) holds.  $\square$

Remark: for  $p=1$  we recover the theorem 2.2 in [JX].

#### 4. THE CURVATURE TENSOR OF IRREDUCIBLE RIEMANNIAN SYMMETRIC SPACES OF NON COMPACT TYPE

Let  $M = G/K$  be a Riemannian symmetric space of non compact type with Lie algebra decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  and Cartan involution  $\theta$ . Let  $\mathfrak{a} \subset \mathfrak{p}$  be a maximal abelian subspace over  $\mathbb{R}$ ,  $rank(M) = dim(\mathfrak{a}) = r$ , we extend  $\mathfrak{a}$  to a Cartan subalgebra of  $\mathfrak{g}$ , say  $\mathfrak{h} = \mathfrak{h}_\theta + \mathfrak{a}$ ,  $\mathfrak{h}_\theta \subset \mathfrak{k}$ . Let  $\Delta = \Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{h}^\mathbb{C})$  be the corresponding root system of complex semisimple Lie algebra  $\mathfrak{g}^\mathbb{C}$ . Let  $\Sigma$  be the restricted root system, i.e, consist of restriction of root of  $\Delta$  to  $\mathfrak{a}$ . For  $\lambda \in \Sigma$ , we denote  $\mathfrak{g}_\lambda$  the root space

with multiplicity  $m_\lambda = \dim \mathfrak{g}_\lambda$ . We have the Iwasawa decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$ . Any vector in  $\mathfrak{p}$  is conjugate by  $\text{Int}(\mathfrak{k})$  to a vector in  $\mathfrak{a}$ . Let  $(X, Y) = B(X, Y)$  be the killing form of  $\mathfrak{g}$  which induces a left invariant inner product on  $M$  by  $\langle X, Y \rangle = -B(X, \theta Y)$ . For any root  $\alpha$ , We embed it into  $\mathfrak{h}$  by  $\alpha(h) = (\alpha, h)$ ,  $h \in \mathfrak{h}$ . The curvature tensor of  $M$  is

$$\begin{aligned} R(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} = -ad[X, Y] \\ R(X, Y, Z, W) &= -\langle R(X, Y)Z, W \rangle \\ &= ([X, Y], [Z, W]) = -\langle [X, Y], [Z, W] \rangle. \end{aligned}$$

Let  $h \in \mathfrak{a}$ ,  $|h| = 1$ ,  $X \perp h$ ,  $X \in \mathfrak{p}$ , then we have

$$\mathcal{R}_h X = R(X, h)h = -[[X, h], h] = -ad^2 h(X).$$

From this, we get the eigenvalues and the eigenvectors as follows,

- (i) 0, the orthogonal complement of  $h$  in  $\mathfrak{a}$ , with multiplicity  $r - 1$ .
- (ii)  $-\lambda^2(h)$ ,  $\mathfrak{g}_\lambda$ , with multiplicity  $m_\lambda$ , where  $\lambda \in \Sigma^+$ .

Since  $|h| = 1$ , we have  $\sum_i \lambda_i^2(h) = \frac{1}{2}$ , which shows that  $\text{Ric} = -\frac{1}{2}$ .

We see that at  $x = \exp_o(r(x)h)$ , the eigenvalues with multiplicity of  $\text{Hess}(r)$  are

$$0, r - 1; \quad |\lambda(h)|, m_\lambda.$$

Now we give the rule for the vanishing of harmonic 1-form in the cases of irreducible symmetric spaces of noncompact type. From (3.2) we see if  $\frac{B}{A} \geq 2$ , then any harmonic 1-form vanishes, where  $B$  is the Ricc curvature and  $A$  is the lower bound of the sectional curvature, this is proved in [JX].

Moreover, from (3.1), if for any restricted root  $\lambda$ , there exists other two restricted roots  $\nu, \mu$  such that

$$(4.1) \quad |\lambda| \leq |\nu| + |\mu|.$$

Then also any harmonic 1-form vanishes.

We note that in condition (4.1) we count the restricted roots with multiplicities.

We are ready to prove Theorem1.4. By Theorem1.3, we only consider one of the following cases,

$$\begin{aligned} SL(3, \mathbb{R})/SO(3), \quad &SU(1, 2)/S(U(1) \times U(2)), \\ SO_0(2, 3)/SO(2) \times SO(3), \quad &Sp(2, \mathbb{R})/U(2). \end{aligned}$$

We adopt the convention from [HEL]. Now we verify the condition (3.1) for these cases.

(1)  $M = SL(3, \mathbb{R})/SO(3)$ , then  $\text{rank}(M) = 2$ ,  $\dim M = 5$ ,  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}_2$ , the restricted root system is also  $\mathfrak{a}_2$ . The positive restricted roots are

$$\lambda_1, \lambda_2, \lambda_1 + \lambda_2.$$

We have

$$|\lambda_1| \leq |\lambda_2| + |\lambda_1 + \lambda_2|, |\lambda_2| \leq |\lambda_1| + |\lambda_1 + \lambda_2|, |\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_2|.$$

Then condition (3.1) is true.

(2)  $M = SU(1, 2)/S(U(1) \times U(2))$ , then  $\text{rank}(M) = 1$ ,  $\dim M = 4$ ,  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}_2$ , the positive restricted roots are

$$\lambda_1, \lambda_1, 2\lambda_1.$$

We have

$$|2\lambda_1| \leq |\lambda_1| + |\lambda_1|.$$

Then condition (3.1) is true.

(3)  $SO_0(2, 3)/SO(2) \times SO(3)$ , then  $\text{rank}(M) = 2$ ,  $\dim(M) = 6$ ,  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{b}_2$ , the positive restricted roots are

$$\lambda_1, \lambda_2, \lambda_1 + \lambda_2, \lambda_1 + 2\lambda_2.$$

From

$$\begin{aligned} |\lambda_1| &\leq |\lambda_2| + |\lambda_1 + \lambda_2|, |\lambda_2| \leq |\lambda_1| + |\lambda_1 + \lambda_2| \\ |\lambda_1 + \lambda_2| &\leq |\lambda_1| + |\lambda_2|, |\lambda_1 + 2\lambda_2| \leq |\lambda_2| + |\lambda_1 + \lambda_2|. \end{aligned}$$

We see that condition (3.1) is true.

(4)  $M = Sp(2, \mathbb{R})/U(2)$ , then  $\text{rank}(M) = 2$ ,  $\dim M = 6$ ,  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{c}_2$ , the positive restricted roots are

$$\lambda_1, \lambda_2, \lambda_1 + \lambda_2, 2\lambda_1 + \lambda_2.$$

From

$$\begin{aligned} |\lambda_1| &\leq |\lambda_2| + |\lambda_1 + \lambda_2|, |\lambda_2| \leq |\lambda_1| + |\lambda_1 + \lambda_2| \\ |\lambda_1 + \lambda_2| &\leq |\lambda_1| + |\lambda_2|, |2\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_1 + \lambda_2|. \end{aligned}$$

We see that condition (3.1) is true.

Now we get Theorem 1.4 from Theorem 1.1 and Theorem 3.1.

**Remark 4.1.** *The only exceptional case is  $SO_0(2, 2)/SO(2) \times SO(2)$ , then  $\text{rank}(M) = 2$ ,  $\dim(M) = 4$ ,  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{d}_2$ , the positive restricted roots are*

$$\lambda_1, \lambda_2, \lambda_1 \perp \lambda_2.$$

*The condition (3.1) is not satisfied.*

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